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## Riemann-Hilbert factorization and Fredholm property of differential operators

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### 1. Introduction

In 1974 Kashiwara-Kawai-Sjöstrand showed the sufficient condition for the convergence of all formal power series solutions for the following linear partial differential operators of regular singular type with analytic coefficients in some neighborhood of the origin of  $\mathbb{C}^n$  ( $n \geq 2$ )

$$\mathcal{L}(x, D) = \sum_{|\alpha|=|\beta| \leq m} x^\alpha a_{\alpha\beta} D^\beta, \quad (1.1)$$

where  $m$  is a positive integer and  $a_{\alpha\beta}$ 's are complex constants. Here we use the standard notations of multi indices,  $D^\beta = (\partial/\partial x_1)^{\beta_1} \cdots (\partial/\partial x_n)^{\beta_n}$  and  $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$  for  $x = (x_1, \dots, x_n) \in \mathbb{C}^n$ . They proved the following result.

**Theorem 1.1.** (cf. [3]) *Suppose that the following condition*

$$\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta} z^\alpha \bar{z}^\beta \neq 0, \quad (1.2)$$

*is satisfied for any  $z \in \mathbb{C}^n \setminus \{0\}$ , where  $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$  and  $\bar{z}^\beta = \bar{z}_1^{\beta_1} \cdots \bar{z}_n^{\beta_n}$ . Then, for any  $f(x)$  analytic at the origin all formal power series solutions  $u(x)$  of the equation  $\mathcal{L}(x, D)u(x) = f(x)$  converges in some neighborhood of the origin.*

They proved the result for somewhat more general operators than (1.1) admitting perturbations. In the following, we call (1.2) as a K-K-S condition.

Inspired from this theorem, we shall study in this paper the Fredholm property in the space of (formal) Gevrey classes for linear partial differential operators with analytic coefficients in a neighborhood of the origin of  $\mathbb{C}^2$ . We consider regular and irregular singular type operators including (1.1). We show that such a property is characterized by the Riemann-Hilbert factorization condition for the Toeplitz symbol on the two dimensional torus in  $\mathbb{C}^2$ . Here the Toeplitz symbols are introduced in a natural way in connection with the filtration with respect to the Gevrey order and it coincides with the symbol given by Kashiwara-Kawai-Sjöstrand in some special cases. (cf. (1.2) and (2.7)). Moreover, we can give an alternative proof of Theorem 1.1 in case  $n = 2$ .

As to the geometrical relations between formal Gevrey spaces and the operators we refer the reader to [9] and [10].

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## 2. Statement of the results

Let  $w_j > 0$  ( $j = 1, 2$ ) and  $s > 0$ . We set  $w = (w_1, w_2)$ . Let  $\mathbf{C}[[x]]$  be the set of all formal power series

$$\mathbf{C}[[x]] := \left\{ u(x); u(x) = \sum_{\eta \in \mathbf{N}^2} u_\eta x^\eta / \eta! \right\}.$$

If we denote by  $\mathcal{O}(\{|x_1| < w_1\} \times \{|x_2| < w_2\})$  the set of holomorphic functions on a domain  $\{|x_1| < w_1\} \times \{|x_2| < w_2\} \subset \mathbf{C}^2$ , then we define the class of entire and Gevrey spaces  $\mathcal{G}_w^s$  by

$$\mathcal{G}_w^s = \left\{ u(x) = \sum_{\eta} u_\eta \frac{x^\eta}{\eta!} \in \mathbf{C}[[x]]; \sum_{\eta} u_\eta \frac{x^\eta}{|\eta|!^s} \in \mathcal{O}(\{|x_1| < w_1\} \times \{|x_2| < w_2\}) \right\}, \quad (2.1)$$

where factorial is understood as the gamma function,  $r! := \Gamma(r+1)$  for  $r \geq 0$ .  $\mathcal{G}_w^s$  can be seen as a Frechét space by the following isomorphism of Frechét spaces

$$\mathbf{C}[[x]] \supset \mathcal{G}_w^s \xrightarrow[\sim]{\text{Borel transf.}} \mathcal{O}(\{|x_1| < w_1\} \times \{|x_2| < w_2\}), \quad (2.2)$$

where the Borel transformation is defined by

$$\mathcal{G}_w^s \ni \sum_{\eta \in \mathbf{N}} u_\eta \frac{x^\eta}{\eta!} \xrightarrow{\sim} \sum_{\eta \in \mathbf{N}} u_\eta \frac{x^\eta}{|\eta|!^s} \in \mathcal{O}(\{|x_1| < w_1\} \times \{|x_2| < w_2\}). \quad (2.3)$$

We note that  $\mathcal{G}_w^s$  is equal to formal Gevrey space, the class of locally analytic functions and entirely analytic functions with finite order in case  $s > 1$ ,  $s = 1$  and  $s < 1$ , respectively. (cf. Lemma 3.1 which follows).

Let  $P \equiv P(x, D_x)$  be a partial differential operator of finite order with holomorphic coefficients in a neighbourhood of the origin of  $\mathbf{C}^2$  and write it in the form,

$$P(x, D_x) = \sum_{\beta \in \mathbf{N}^2, |\beta| \leq m} a_\beta(x) D^\beta, \quad (2.4)$$

where  $a_\beta(x)$  is an analytic function of  $x$  in some neighborhood of the origin.

By substituting the Taylor expansion of  $a_\beta(x)$ ,  $a_\beta(x) = \sum_{\alpha} a_{\alpha\beta} x^\alpha$  in (2.4) we have the expression

$$P(x, D) = \sum_{\alpha, \beta} a_{\alpha\beta} x^\alpha D^\beta. \quad (2.5)$$

For  $x^\alpha D^\beta$  we define the  $s$ -Gevrey order of  $x^\alpha D^\beta$  by

$$\text{ord}_s x^\alpha D^\beta := |\beta| + (1-s)(|\alpha| - |\beta|). \quad (2.6)$$

Then the  $s$ -Gevrey order of  $P$  in (2.5) is defined by

$$\text{ord}_s P := \sup_{\alpha, \beta} \{ |\beta| + (1-s)(|\alpha| - |\beta|) ; a_{\alpha\beta} \neq 0 \}.$$

Here and in what follows we always assume that the  $s$ -Gevrey order of  $P(x, D)$  is finite. This implies that  $P$  is of polynomial coefficients in case  $s < 1$ .

We shall define the Toeplitz symbol associated with  $P(x, D)$  by

$$L_s(z; \xi) := \sum_{|\beta| + (1-s)(|\alpha| - |\beta|) = \text{ord}_s P} a_{\alpha\beta} z^{\alpha - \beta} w^{\alpha - \beta} \xi^\alpha, \quad \xi \in \mathbb{R}^2. \quad (2.7)$$

We define the two dimensional torus  $\mathbb{T}^2$  by  $\mathbb{T}^2 = \{(z_1, z_2) \in \mathbb{C}^2; |z_1| = 1, |z_2| = 1\}$ . Then we can prove the following

**Theorem 2.1.** *The operator  $P : \mathcal{G}_w^s \longrightarrow \mathcal{G}_w^s$  is Fredholm of index zero in the sense that the mapping has the same finite dimensional kernel and cokernel if the following conditions are satisfied.*

$$L_s(z, \xi) \neq 0 \quad \forall (z_1, z_2) \in \mathbb{T}^2, \forall \xi \in \mathbb{R}^2, |\xi| = 1, \xi \geq 0. \quad (2.8)$$

$$\text{ind}_1 L_s = \text{ind}_2 L_s = 0. \quad (2.9)$$

Here  $\text{ind}_1 L_s$  (resp.  $\text{ind}_2 L_s$ ) is defined by

$$\text{ind}_1 L_s = \frac{1}{2\pi i} \oint_{|\zeta_1|=1} d \log L_s(\zeta_1, z_2, \eta). \quad (2.10)$$

*Remarks.* (a) The conditions (2.8) and (2.9) are equivalent to a Riemann-Hilbert factorization condition with respect to a certain closed subspace  $G_{w,\xi}^s(\mu)$  of  $G_w^s(\mu)$  which is isomorphic to the Hardy space and will be defined in the proof of Theorem 2.1. We have to note that these conditions are necessary and sufficient condition for the Fredholmness of a Toeplitz operator  $T_\xi$  on  $G_{w,\xi}^s(\mu)$  which will be reduced from the mapping  $P : \mathcal{G}_w^s \rightarrow \mathcal{G}_w^s$ . (cf. Theorem 4.1.)

(b) We note that the right-hand side of (2.10) is an integer-valued continuous function of  $z_2$  and  $\eta$ . Because the sets  $|z_2| = 1$  and  $|\eta| = 1$  are connected the integral (2.10) is constant. Hence the right-hand side is independent of  $z_2$  and  $\eta$ . We write this quantity by  $\text{ind}_1 L_s$ . We similarly define  $\text{ind}_2 L_s$ .

**Corollary 2.2.** *Suppose that the K-K-S condition (1.2) is satisfied. Assume that  $n = 2$ . Then the operator  $\mathcal{L} : \mathcal{G}_w^1 \longrightarrow \mathcal{G}_w^1$  is a Fredholm operator of index zero. Especially, for  $f(x)$  analytic in a neighbourhood of the origin, if formal power series  $f(x)$  satisfies  $P(x, D)u(x) = f(x)$  then  $u(x)$  is also analytic in a neighbourhood of the origin.*

*Proof.* We shall show that (2.8) and (2.9) with  $s = 1$ . Let  $p_K := \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta} z^\alpha \bar{z}^\beta$ . Suppose that  $\xi_1 \xi_2 \neq 0$ . If we make the change of variables  $x \mapsto wx = (w_1 \xi_1^{1/2} x_1, \xi_2^{1/2} w_2 x_2)$  in the original equation (1.1), then the symbol  $p_K$  in (1.2) can be replaced by

$$\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta} z^\alpha \bar{z}^\beta w^{\alpha - \beta} \xi^{\alpha/2 - \beta/2}. \quad (2.11)$$

If we set  $z \mapsto \xi^{1/2}z$  in (2.11) it follows that

$$L_s(z, \xi) = p_K, \quad (2.12)$$

if  $z = (z_1, z_2) \in \mathbf{T}^2$ , and  $\xi \in \mathbf{R}^2$  satisfies that  $|\xi| = 1, \xi \geq 0, \xi \neq (1, 0)$  and  $(0, 1)$ . Because both sides of (2.12) are analytic functions of  $z$  in some neighborhood of  $\mathbf{T}^2$  and  $\xi$  in some neighborhood of  $\xi \in \mathbf{R}^2, |\xi| = 1, \xi \geq 0$  (2.12) holds in case  $\xi = (1, 0)$  or  $\xi = (0, 1)$ . Therefore, (2.8) is direct consequence of a K-K-S condition. On the other hand, if we make the deformation of a path  $z_1 \mapsto \varepsilon z_1, z_2 \neq 0$  ( $\varepsilon > 0$ ) or  $z_2 \mapsto \varepsilon z_2, z_1 \neq 0$  ( $\varepsilon > 0$ ) we have (2.9). Hence, by Theorem 2.1,  $\mathcal{L}$  is a Fredholm operator (with an index zero) on  $\mathcal{G}_w^1$  for every  $w > 0$ .

Next, let assume for a formal power series  $u(x) \in \mathbf{C}[[x]]$  it holds that  $\mathcal{L}u(x) = f(x)$  is analytic in a neighbourhood of the origin. We may assume  $f(x) \in \mathcal{G}_w^1$  for some  $w > 0$ . Let  $u_n(x)$  be a homogeneous polynomial of degree  $n$ . Then  $\mathcal{L}u_n(x)$  is also homogeneous of degree  $n$  by the definition of operator  $\mathcal{L}$ . Hence, the basis of kernel of the mapping  $\mathcal{L} : \mathcal{G}_w^1 \rightarrow \mathcal{G}_w^1$  consists of finite numbers of homogeneous polynomials. Therefore for homogeneous polynomial  $f_n(x)$  of degree  $n$  of sufficiently large  $n$ , there exists a unique homogeneous polynomial  $u_n(x)$  of degree  $n$  satisfying  $\mathcal{L}u_n(x) = f_n(x)$ , since the uniqueness of solutions implies the solvability. This implies that the mapping  $\mathcal{L} : \mathbf{C}[[x]] \rightarrow \mathbf{C}[[x]]$  is also a Fredholm operator (with an index zero) and has the same dimensional kernel and cokernel with the operator  $\mathcal{L}$  on  $\mathcal{G}_w^s$ . Therefore, the mapping

$$\mathcal{L} : \mathbf{C}[[x]]/\mathcal{G}_w^1 \longrightarrow \mathbf{C}[[x]]/\mathcal{G}_w^1$$

is a Fredholm operator with an index zero. The above reasoning shows that this mapping is surjective, and therefore is injective. This proves the latter half of the corollary.

### 3. Preliminary lemmas

We define the class  $G_w^s(\mu)$  ( $\mu \in \mathbf{R}$ ) by

$$G_w^s(\mu) := \left\{ u = \sum_{\eta} u_{\eta} \frac{x^{\eta}}{\eta!}; \sum_{\eta} \left( |u_{\eta}| \frac{w^{\eta}}{(|\eta| - \frac{\mu}{s})!^s} \right)^2 < \infty \right\}, \quad (3.1)$$

where factorial is understood as the gamma function,  $r! := \Gamma(r+1)$  for  $r \geq 0$  and where we set  $(|\eta| - (\mu/s))! = 1$  if  $|\eta| - (\mu/s) \leq 0$ .  $G_w^s(\mu)$  is a Banach space with the norm

$$\|u\| := \left( \sum_{\eta} \left( |u_{\eta}| \frac{w^{\eta}}{(|\eta| - \frac{\mu}{s})!^s} \right)^2 \right)^{1/2}.$$

**Lemma 3.1.** *Let the class  $\mathcal{G}_w^s$  be defined by (2.1). Then we have*

$$\mathcal{G}_w^s = \text{proj} \lim_{r \uparrow w} G_r^s(\mu) \quad (3.2)$$

for every  $\mu \in \mathbf{R}$ .

*Proof.* Suppose that  $u(x) \in G_r^s(\mu)$  for any  $r < w$ . Then we have  $|u_\eta| \leq Mr^{-\eta}(|\eta| - (\mu/s))!^s$  for some  $M > 0$  independent of  $\eta$ . Therefore we have, for  $|x_j| < r_j$ ,

$$\sum |u_\eta| \frac{|x|^\eta}{|\eta|!^s} \leq M \sum r^{-\eta} |x^\eta| \frac{(|\eta| - \frac{\mu}{s})!^s}{|\eta|!^s}.$$

Clearly, the right-hand side converges for  $|x| < r$ . Because  $r < w$  is arbitray we have  $u \in \mathcal{G}_w^s$ .

Conversely, suppose that  $u = \sum u_\eta x^\eta / \eta! \in \mathcal{G}_w^s$ . Then we have  $U(x) := \sum_\eta u_\eta x^\eta / |\eta|!^s \in \mathcal{O}(\{|x_1| \leq \rho_1\} \times \{|x_2| \leq \rho_2\})$  for any  $\rho < w$ . By Cauchy's formula we have

$$v_\eta := \frac{u_\eta}{|\eta|!^s} = \frac{1}{(2\pi i)^2} \oint_{|\zeta_1|=\rho_1} \oint_{|\zeta_2|=\rho_2} \frac{U(\xi)}{\xi^{\eta+1}} d\xi.$$

Hence we have the estimate  $|v_\eta| \leq M\rho^{-\eta}$  for some  $M > 0$ . Because  $\rho < w$  is arbitrary we have  $u \in G_r^s(\mu)$  for any  $r < w$ .  $\square$

Let  $X_j$  ( $j = 1, 2$ ) be a positive number and set  $X = (X_1, X_2)$ . We denote by  $\mathcal{O}(|x| \leq X)$  the set of holomorphic functions on  $\{x \in \mathbf{C}^2; |x_j| < X_j, j = 1, 2\}$  and continuous on its closure. For  $a(x) \in \mathcal{O}(|x| \leq X)$ , we put  $\|a\|_X := \max_{|x| \leq X} |a(x)|$ . Then we have

**Lemma 3.2.** *Let  $s \geq 1$ . Assume that  $a(x) \in \mathcal{O}(|x| \leq \rho w)$  ( $\rho > 1$ ), then for any  $U(x) \in G_w^s(\mu)$ , we have  $a(x)U(x) \in G_w^s(\mu)$  and there exists a constant  $C$  depending only on  $\mu$  such that*

$$\|aU\| \leq C \left( \frac{\rho}{\rho-1} \right)^2 \|a\|_{\rho w} \|U\|. \quad (3.3)$$

*Proof.* We put  $a(x) = \sum a_\gamma x^\gamma / \gamma! \in \mathcal{O}(|x| \leq \rho w)$ . Then by Cauchy's integral formula, we have  $|a_\gamma| \leq \|a\|_{\rho w} \gamma! / (\rho w)^\gamma$  ( $\gamma \in \mathbf{N}^2$ ). We put  $a(x)U(x) = \sum V_\beta x^\beta / \beta!$ . Then we have

$$V_\beta = \sum_{0 \leq \gamma \leq \beta} a_\gamma U_{\beta-\gamma} \frac{\beta!}{(\beta-\gamma)! \gamma!}.$$

Hence we have, for  $C_1 > 0$

$$\begin{aligned} \sum_\beta \left( |V_\beta| \frac{w^\beta}{(|\beta| - \frac{\mu}{s})!^s} \right)^2 &\leq \|a\|_{\rho w}^2 \sum_\beta \left( \sum_{0 \leq \gamma \leq \beta} |U_{\beta-\gamma}| \frac{1}{(\rho w)^\gamma} \frac{\beta!}{(\beta-\gamma)!} \frac{w^\beta}{(|\beta| - \frac{\mu}{s})!^s} \right)^2 \\ &\leq C_1 \|a\|_{\rho w}^2 \sum_\beta \left( \sum_{0 \leq \gamma \leq \beta} |U_{\beta-\gamma}| \frac{1}{\rho^{|\gamma|}} \frac{w^{\beta-\gamma}}{(|\beta| - \frac{\mu}{s} - |\gamma|)!^s} \right)^2 \\ &\leq C_1 \|a\|_{\rho w}^2 \sum_\beta \left( \sum_\gamma \frac{1}{\rho^{|\gamma|}} \right) \left( \sum_\gamma \frac{1}{\rho^{|\gamma|}} \left( |U_{\beta-\gamma}| \frac{w^{\beta-\gamma}}{(|\beta| - \frac{\mu}{s} - |\gamma|)!^s} \right)^2 \right) \end{aligned}$$

$$\begin{aligned}
&\leq C_1 \left( \frac{\rho}{\rho-1} \right)^2 \|a\|_{\rho w}^2 \sum_{\gamma} \frac{1}{\rho^{|\gamma|}} \sum_{\beta \geq \gamma} \left( |U_{\beta-\gamma}| \frac{w^{\beta-\gamma}}{(|\beta-\gamma| - \frac{\mu}{s})!^s} \right)^2 \\
&\leq \left( \frac{\rho}{\rho-1} \right)^4 C_1 \|a\|_{\rho w}^2 \|U\|^2. \quad \square
\end{aligned}$$

**Lemma 3.3.** Let  $\mu = |\beta| + (1-s)(|\alpha| - |\beta|)$  be the  $s$ -Gevrey order of  $x^\alpha D^\beta$ . Then the map  $x^\alpha D^\beta : G_w^s(\mu) \longrightarrow G_w^s(0)$  is continuous. Moreover, for every  $\varepsilon > 0$  the map  $x^\alpha D^\beta : G_w^s(\mu + \varepsilon) \longrightarrow G_w^s(0)$  is a compact operator.

*Proof.* We first show that for every  $\kappa < \mu$  the injection  $i; G_w^s(\mu) \rightarrow G_w^s(\kappa)$  is compact. Let  $B \subset G_w^s(\mu)$  be a bounded set of  $G_w^s(\mu)$ . If we write  $u = \sum_{\eta} u_{\eta} x^{\eta} / \eta! \in B$ , then for each fixed  $\eta$  the set  $\{u_{\eta}; u \in B\}$  is bounded. Hence, by the diagonal argument, we can choose a sequence  $\{u^{(k)}\} \subset B$ ,  $u^{(k)}(x) = \sum_{\eta} u_{\eta}^{(k)} x^{\eta} / \eta!$  such that for each  $\eta$ ,  $u_{\eta}^{(k)} \rightarrow u_{\eta}$  when  $k \rightarrow \infty$ . Moreover we have that

$$\begin{aligned}
\sum_{|\eta| \geq N} \left( |u_{\eta}^{(k)}| \frac{w^{\eta}}{(|\eta| - \frac{\kappa}{s})!^s} \right)^2 &\leq \max_{|\eta| \geq N} \frac{(|\eta| - \frac{\mu}{s})!^{2s}}{(|\eta| - \frac{\kappa}{s})!^{2s}} \sum_{|\eta| \geq N} \left( |u_{\eta}^{(k)}| \frac{w^{\eta}}{(|\eta| - \frac{\mu}{s})!^s} \right)^2 \\
&\leq K \max_{|\eta| \geq N} \frac{(|\eta| - \frac{\mu}{s})!^{2s}}{(|\eta| - \frac{\kappa}{s})!^{2s}} \longrightarrow 0 \quad (n \rightarrow \infty),
\end{aligned}$$

where  $K > 0$  is independent of  $k$  and  $N$ . This proves that the sequence  $\{u^{(k)}\}$  converges in  $G_w^s(\mu)$ .

In order to complete the proof we shall show that the map  $x^\alpha D^\beta : G_w^s(\mu) \longrightarrow G_w^s(0)$  is continuous. By simple calculations

$$x^\alpha D^\beta \sum_{\eta} u_{\eta} \frac{x^{\eta}}{\eta!} = \sum_{\eta} u_{\eta} \frac{x^{\eta+\alpha-\beta}}{(\eta-\beta)!} = \sum_{\eta} u_{\eta+\beta-\alpha} \frac{x^{\eta}}{(\eta-\alpha)!}. \quad (3.4)$$

Hence we have

$$\sum_{\eta} \left( |u_{\eta+\beta-\alpha}| \frac{w^{\eta}}{|\eta|!^s} \frac{\eta!}{(\eta-\alpha)!} \right)^2 = \sum_{\eta} \left( |u_{\eta}| w^{\eta-\beta+\alpha} \frac{1}{(|\eta| - |\beta| + |\alpha|)!^s} \frac{(\eta - \beta + \alpha)!}{(\eta - \beta)!} \right)^2. \quad (3.5)$$

If  $\eta$  is sufficiently large the term  $(\eta - \beta + \alpha)! / (\eta - \beta)!$  can be estimated from the above and from the below by  $\eta^{\alpha}$ . Therefore we have

$$\frac{(|\eta| - \frac{\mu}{s})!^s}{(|\eta| - |\beta| + |\alpha|)!^s} \frac{(\eta - \beta + \alpha)!}{(\eta - \beta)!} \leq C |\eta|^{s(|\beta| - |\alpha|) - \mu} |\eta|^{|\alpha|} = C |\eta|^{s(|\beta| - |\alpha|) + |\alpha| - \mu} \quad (3.6)$$

for some constant  $C$  is independent of  $\eta$ . Because  $s(|\beta| - |\alpha|) + |\alpha| - \mu = 0$  the right-hand side of (3.6) is bounded when  $|\eta|$  tends to infinity. By (3.4), (3.5) and (3.6) we see that the map  $x^\alpha D^\beta : G_w^s(\mu) \longrightarrow G_w^s(0)$  is continuous.  $\square$

Let  $p(\eta)$  be a function on  $\mathbb{N}^2$  such that

$$|p(\eta)| \leq C|\eta|^m, \forall \eta \in \mathbb{N}^2 \quad (3.7)$$

for some  $C > 0$  and  $m \geq 0$  independent of  $\eta$ . Then we define the Euler type pseudodifferential operator  $p(\partial)$  on  $G_w^s(\mu)$  by

$$p(\partial)u := \sum_{\eta} u_{\eta} p(\eta) x^{\eta} / \eta!, \quad u = \sum_{\eta} u_{\eta} x^{\eta} / \eta! \in G_w^s(\mu), \quad (3.8)$$

where we set  $\partial = (\partial_1, \partial_2)$ ,  $\partial_j = x_j(\partial/\partial x_j)$ ,  $j = 1, 2$ . We note that if  $p(\eta) = \eta_1 + \eta_2$ , then  $p(\partial) = \partial_1 + \partial_2$  is a so-called Euler type differential operator. Then we have

**Lemma 3.4.** *Let  $p(\eta)$  be a function on  $\mathbb{N}^2$  such that  $\sup_{|\eta| \geq N} |p(\eta)| \rightarrow 0$  when  $N \rightarrow \infty$ . Then the map  $p(\partial) : G_w^s(\mu) \rightarrow G_w^s(\mu)$  is a compact operator for every  $\mu$ .*

The proof of this lemma follows exactly the same arguments of the former half of the proof of Lemma 3.3. Therefore we omit the proof.

#### 4. Proof of Theorem 2.1.

Let  $m$  be an  $s$ -Gevrey order of  $P$ . In view of Lemma 3.1 it is sufficient to prove that for any  $r < w$  the map  $P : G_r^s(m) \rightarrow G_r^s(0)$  is Fredholm of index 0. For every  $\beta$  we collect  $\alpha \in \mathbb{N}^2$  such that  $|\beta| + (1-s)(|\alpha| - |\beta|) < m$  and we denote the set by  $C_{\beta}$ . Since  $C_{\beta}$  is a subset of  $\mathbb{N}^2$ , we can choose finite  $\alpha^{(j)}$ 's ( $j = 1, \dots, k$ ) from  $C_{\beta}$  such that  $C_{\beta}$  is contained in the union of sets  $\alpha^{(j)} + \mathbb{N}^2$  for  $j = 1, \dots, k$ . We choose the set of  $\alpha^{(j)}$ 's ( $j = 1, \dots, k$ ) for each  $\beta$ . By using this grouping of  $\alpha$  we can write  $P$  in (2.5) in the following form

$$\begin{aligned} P(x, D_x) &= \sum_{|\beta| + (1-s)(|\alpha| - |\beta|) = m} a_{\alpha\beta} x^{\alpha} D^{\beta} + \sum_{|\beta| + (1-s)(|\alpha| - |\beta|) < m, \alpha, \text{finite}} a_{\alpha\beta} x^{\alpha} D^{\beta} \\ &+ \sum_{|\beta| + (1-s)(|\alpha| - |\beta|) < m, \alpha, \text{finite}} b_{\alpha\beta}(x) x^{\alpha} D^{\beta} =: P_0(x, D) + P_1(x, D) + P_2(x, D), \end{aligned} \quad (4.1)$$

where  $a_{\alpha\beta}$  are complex constants and  $b_{\alpha\beta}(x)$  are analytic functions of  $x$ . We note that  $P_2(x, D) \equiv 0$  if  $s < 1$  by assumption.

Because the  $s$ -Gevrey order of terms in  $P_1$  is small than  $m$ , it follows from Lemma 3.3 that the map  $P_1 : G_w^s(m) \rightarrow G_w^s(0)$  is compact. On the other hand, since  $s \geq 1$ , it follows from Lemmas 3.2 and 3.3 that the map  $P_2 : G_w^s(m) \rightarrow G_w^s(0)$  is compact. Therefore we shall consider the Fredholmness of  $P_0 : G_w^s(m) \rightarrow G_w^s(0)$ .

By using the identity of ordinary differential equations

$$t^k \frac{d^k}{dt^k} = t \frac{d}{dt} \left( t \frac{d}{dt} - 1 \right) \cdots \left( t \frac{d}{dt} - k + 1 \right), \quad k = 1, 2, \dots,$$

we have

$$x^{\beta} \left( \frac{\partial}{\partial x} \right)^{\beta} = \prod_{j=1}^2 x_j \frac{\partial}{\partial x_j} \left( x_j \frac{\partial}{\partial x_j} - 1 \right) \cdots \left( x_j \frac{\partial}{\partial x_j} - \beta_j + 1 \right) =: p_{\beta}(\partial). \quad (4.2)$$



By substituting (4.2) into (4.1) we have

$$P_0(x, D) = \sum_{|\beta|+(1-s)(|\alpha|-|\beta|)=m} a_{\alpha\beta} x^\alpha D^\beta = \sum_{|\beta|+(1-s)(|\alpha|-|\beta|)=m} a_{\alpha\beta} x^{\alpha-\beta} p_\beta(\partial). \quad (4.3)$$

We set  $\langle \eta \rangle := (1 + |\eta|^2)^{1/2}$  and we denote by  $\langle \partial \rangle$  the Euler type pseudodifferential operator with symbol  $\langle \eta \rangle$  given by (3.8).

Let  $\gamma \in \mathbb{Z}^2$  and let  $u = \sum u_\eta x^\eta / \eta! \in G_w^s(\mu)$ . We set  $v_\eta := u_\eta w^\eta / (|\eta| - (\mu/s))!^s$ . Then we have

$$\begin{aligned} x^\gamma \langle \partial \rangle^{(s-1)|\gamma|} \sum u_\eta \frac{x^\eta}{\eta!} &= x^\gamma \langle \partial \rangle^{(s-1)|\gamma|} \sum v_\eta w^{-\eta} (|\eta| - \frac{\mu}{s})!^s \frac{x^\eta}{\eta!} \\ &= \sum v_\eta w^{-\eta} (|\eta| - \frac{\mu}{s})!^s \langle \eta \rangle^{(s-1)|\gamma|} \frac{x^{\eta+\gamma}}{\eta!} \\ &= \sum v_{\eta-\gamma} w^{-\eta+\gamma} (|\eta| - |\gamma| - \frac{\mu}{s})!^s \langle \eta - \gamma \rangle^{(1-s)|\gamma|} \frac{\eta!}{(\eta - \gamma)!} \frac{x^\eta}{\eta!}. \end{aligned}$$

On the other hand, we see that  $u \in G_w^s(\mu)$  if and only if the sequence  $\{v_\eta\}$  is in  $\ell_2 := \ell_2(\mathbb{Z}^2)$ , the set of square summable sequences on  $\mathbb{Z}^2$ , where we set  $v_\eta = 0$  if  $\eta \notin \mathbb{N}^2$ . By this identification,  $x^\gamma \langle \partial \rangle^{(s-1)|\gamma|}$  induces the map

$$x^\gamma \langle \partial \rangle^{(s-1)|\gamma|} : \{v_\eta\} \in \ell_2 \longrightarrow \left\{ v_{\eta-\gamma} w^\gamma \frac{(|\eta| - |\gamma| - (\mu/s))!^s}{(|\eta| - (\mu/s))!^s} \langle \eta - \gamma \rangle^{(s-1)|\gamma|} \frac{\eta!}{(\eta - \gamma)!} \right\} \in \ell_2. \quad (4.4)$$

If we set  $\xi = \eta/|\eta|$  and if we let  $|\eta|$  tends to infinity we see that

$$\frac{(|\eta| - |\gamma| - (\mu/s))!^s}{(|\eta| - (\mu/s))!^s} \langle \eta - \gamma \rangle^{(s-1)|\gamma|} \frac{\eta!}{(\eta - \gamma)!} = \lambda_\gamma(\eta) + r_\gamma(\eta), \quad (4.5)$$

where  $\lambda_\gamma(\eta) = \xi^\gamma$ ,  $\xi = \eta/|\eta|$  and where  $r_\gamma(\eta)$  consists of term such that  $r_\gamma(\eta) \rightarrow 0$  when  $\eta$  tends to infinity. We define the shift operator  $S_\gamma$  by

$$S_\gamma : \{v_\eta\}_\eta \in \ell_2 \longrightarrow \{v_{\eta-\gamma}\}_\eta \in \ell_2. \quad (4.6)$$

It follows from (4.4), (4.5) and (4.6) that

$$x^\gamma \langle \partial \rangle^{(s-1)|\gamma|} = S_\gamma \lambda_\gamma(\partial) w^\gamma + R_\gamma(\partial) \quad (4.7)$$

where  $R_\gamma(\partial) := S_\gamma r_\gamma(\partial) w^\gamma$ .

Next we consider the operator  $\langle \partial \rangle^{-|\beta|} p_\beta(\partial)$ . Because we have

$$\langle \partial \rangle^{-|\beta|} p_\beta(\partial) \sum u_\eta \frac{x^\eta}{\eta!} = \sum v_\eta w^{-\eta} (|\eta| - \frac{\mu}{s})!^s \langle \eta \rangle^{-|\beta|} p_\beta(\eta) \frac{x^\eta}{\eta!},$$

$\langle \partial \rangle^{-|\beta|} p_\beta(\partial)$  induces the map

$$\langle \partial \rangle^{-|\beta|} p_\beta(\partial) : \{v_\eta\} \in \ell_2 \longrightarrow \{v_\eta \langle \eta \rangle^{-|\beta|} p_\beta(\eta)\} \in \ell_2. \quad (4.8)$$

We note that

$$\langle \eta \rangle^{-|\beta|} p_\beta(\eta) = \xi^\beta + \tilde{r}(\eta), \quad (4.9)$$

where  $\tilde{r}(\eta)$  satisfies that  $\sup_{|\eta| \geq n} |\tilde{r}(\eta)| \rightarrow 0$  when  $\eta$  tends to infinity. Therefore we have

$$\langle \partial \rangle^{-|\beta|} p_\beta(\partial) = \lambda_\beta(\partial) + \tilde{r}(\partial). \quad (4.10)$$

We set

$$Q_0 := P_0 \langle \partial \rangle^{-m}. \quad (4.11)$$

Clearly,  $Q_0 : G_w^s(\mu) \longrightarrow G_w^s(\mu)$  is bounded for every  $\mu$ . We want to show that  $Q_0$  is Fredholm if and only if  $P_0 : G_w^s(\mu) \longrightarrow G_w^s(\mu + m)$  is Fredholm. Indeed, let  $R_j$  and the compact operator  $K_j (j = 1, 2)$  satisfy that

$$R_1 Q_0 = I + K_1, \quad Q_0 R_2 = I + K_2, \quad (4.12)$$

where  $I$  denotes the identity operator. Let us for the moment suppose that

$$Q_0 = \langle \partial \rangle^{-m} P_0 + K \quad (4.13)$$

for some compact operator  $K$ . Because the map  $\langle \partial \rangle^{-m} : G_w^s(\mu) \longrightarrow G_w^s(\mu + m)$  is bijective it follows from (4.12) and (4.13) that  $P_0$  is a Fredholm operator. We can similarly prove the converse.

It remains to prove (4.13). In view of (4.3), we shall consider the commutator

$$[x^\gamma, \langle \partial \rangle^{-m}] := x^\gamma \langle \partial \rangle^{-m} - \langle \partial \rangle^{-m} x^\gamma,$$

where  $\gamma = \alpha - \beta$ . We have, for  $u = \sum_\eta u_\eta x^\eta / \eta! \in G_w^s(\mu)$

$$[x^\gamma, \langle \partial \rangle^{-m}] u = x^\gamma \sum_\eta u_\eta x^\eta \left( \langle \eta \rangle^{-m} - \langle \gamma + \eta \rangle^{-m} \right) / \eta!. \quad (4.14)$$

By Taylor's formula we have

$$\langle \eta \rangle^{-m} - \langle \gamma + \eta \rangle^{-m} = m \int_0^1 \gamma \cdot (\eta + s\gamma) \langle \eta + s\gamma \rangle^{-m-2} ds =: C_\gamma(\eta). \quad (4.15)$$

It follows that  $\Lambda_\gamma(\eta) := \langle \eta \rangle^m C_\gamma(\eta)$  satisfies

$$\sup_{|\eta| \geq N} \langle \gamma \rangle^{-1} |\Lambda_\gamma(\eta)| \rightarrow 0, \quad (4.16)$$

when  $N \rightarrow \infty$ . Here the limit is uniform with respect to  $\gamma$  when  $|\gamma| \rightarrow \infty$ . Therefore we have

$$[x^\gamma, \langle \partial \rangle^{-m}] = x^\gamma \langle \partial \rangle^{-m} \Lambda_\gamma(\partial) \quad (4.17)$$

where  $\Lambda_\gamma(\partial)$  is the Euler type operator with symbol given by  $\Lambda_\gamma(\eta)$ . It follows from (4.16), (4.17) and Lemma 3.4 that  $[x^\gamma, <\partial>^{-m}]$  is a compact operator. In order to show (4.13) we note that the summation in  $P_0$  with respect to  $\alpha$  and  $\beta$  is a finite sum in case  $s \neq 1$ . Hence the assertion is trivial. If  $s = 1$ , one may assume that  $|a_{\alpha\beta}| \leq C\rho^{|\alpha|}$  for some  $\rho < 1$  and  $C > 0$  independent of  $\alpha$  and  $\beta$ . Because  $a_{\alpha\beta}$  is the Taylor coefficients of  $a_\beta(x)$ , which is, by a scaling of  $x$ , analytic in a larger domain. This implies that we can assume  $\rho < 1$ . Therefore it follows from (4.16) and lemma 3.4 that

$$\sum_{\alpha, \beta} a_{\alpha\beta} [x^{\alpha-\beta}, <\partial>^{-m}] p_\beta(\partial) = \sum_{\alpha, \beta} a_{\alpha\beta} x^{\alpha-\beta} <\alpha - \beta> <\partial>^{-m} p_\beta(\partial) <\alpha - \beta>^{-1} \Lambda_{\alpha-\beta}(\partial)$$

is a compact operator.

Next we want to show that the Fredholmness of the operator  $Q_0$  given by (4.10) is equivalent to that of a certain Toeplitz operator. To this end, we define the projection  $\pi$  by

$$\pi u := \sum_{\eta \in \mathbb{N}^2} u_\eta x^\eta / \eta! \quad \text{for} \quad u = \sum_{\eta \in \mathbb{Z}^2} u_\eta x^\eta / \eta!. \quad (4.18)$$

It follows from the definition of  $p_\beta(\partial)$  in (4.2) that  $x^{\alpha-\beta} p_\beta(\partial) <\partial>^{-m}$  maps  $G_w^s(\mu)$  into itself. Hence we have

$$Q_0 = \pi \sum a_{\alpha\beta} x^{\alpha-\beta} p_\beta(\partial) <\partial>^{-m} = \pi \sum a_{\alpha\beta} x^{\alpha-\beta} <\partial>^{(s-1)(|\alpha|-|\beta|)} <\partial>^{-|\beta|} p_\beta(\partial), \quad (4.19)$$

where the summation is taken for  $\alpha$  and  $\beta$  such that  $|\beta| + (1-s)(|\alpha| - |\beta|) = m$ . By substituting (4.7) with  $\gamma = \alpha - \beta$  and (4.10) into (4.19) we have

$$\begin{aligned} Q_0 &= \pi \sum a_{\alpha\beta} (S_{\alpha-\beta} \lambda_{\alpha-\beta} w^{\alpha-\beta} + R_{\alpha-\beta}) (\lambda_\beta + \tilde{r}) = \pi \sum a_{\alpha\beta} S_{\alpha-\beta} w^{\alpha-\beta} \lambda_{\alpha-\beta} \lambda_\beta \\ &\quad + \pi \sum K_{\alpha\beta} = \pi \sum a_{\alpha\beta} S_{\alpha-\beta} \lambda_\alpha w^{\alpha-\beta} + \pi \sum a_{\alpha\beta} K_{\alpha\beta}, \end{aligned} \quad (4.20)$$

where

$$K_{\alpha\beta} = S_{\alpha-\beta} \lambda_{\alpha-\beta} w^{\alpha-\beta} \tilde{r} + R_{\alpha-\beta} \lambda_\beta + R_{\alpha-\beta} \tilde{r}. \quad (4.21)$$

We want to show that the second term in the right-hand side of (4.20) is a compact operator. To this end, we note that for each  $\alpha$  and  $\beta$ ,  $K_{\alpha\beta}$  is a compact operator by the definition of the symbols  $R_{\alpha-\beta}$  and  $\tilde{r}$  and Lemma 3.4. Because the sum in the second term in the right-hand side of (4.19) is a finite sum except for the case  $s = 1$ , the desired compactness follows if  $s \neq 1$ . In case  $s = 1$  we may assume that  $|a_{\alpha\beta}| \leq C\rho^{|\alpha|}$  for some  $C > 0$  and  $0 < \rho < 1$  without loss of generality. In view of (4.21), it is sufficient to show that the sum  $\sum_{\alpha, \beta} a_{\alpha\beta} R_{\alpha-\beta}$  is a compact operator. In view of the definition of  $R_{\alpha-\beta}$  in (4.7) ( $\gamma = \alpha - \beta$ ), the inequality,  $|(|\eta| - |\gamma|)! \eta! / (|\eta|! (\eta - \gamma)!)| \leq 1$  and (4.4) the symbol  $a_{\alpha, \beta} r_{\alpha-\beta}(\eta)$  tends to zero uniformly with respect to  $\alpha$  and  $\beta$  when  $\eta \rightarrow \infty$ . Hence we have the assertion by Lemma 3.4.

The operator

$$T := \pi \sum_{|\beta| + (1-s)(|\alpha| - |\beta|) = m} a_{\alpha\beta} w^{\alpha-\beta} S_{\alpha-\beta} \lambda_\alpha(\partial) : G_w^s(\mu) \longrightarrow G_w^s(\mu) \quad (4.22)$$

is called a Toeplitz operator. Here  $S_{\alpha-\beta}$  is considered as an operator on  $G_w^s(\mu)$  by the isomorphism between  $\ell^2$  and  $G_w^s(\mu)$ . The function (2.7) is called the symbol of a Toeplitz operator  $T$ .

It follows from (4.19) that  $Q_0$  is a Fredholm operator if and only if the Toeplitz operator  $T$  is a Fredholm operator. By the obvious identification between  $G_w^s(\mu)$  and  $\ell_2(\mathbf{N}^2)$  used in (4.4) one may think  $T$  as an operator on  $\ell_2(\mathbf{N}^2)$ . Moreover, by the isomorphism between  $\ell_2(\mathbf{N}^2)$  and the Hardy space  $H^2(\mathbf{T}^2)$  one can also think  $T$  as an operator on  $H^2(\mathbf{T}^2)$ . If we take the coordinate  $(e^{i\theta_1}, e^{i\theta_2}) \in \mathbf{T}^2$   $T$  is given by

$$T = \pi \sum_{|\beta|+(1-s)(|\alpha|+|\beta|)=m} a_{\alpha\beta} w^{\alpha-\beta} e^{i(\alpha-\beta)\theta} \lambda_\alpha(D) : H^2(\mathbf{T}^2) \longrightarrow H^2(\mathbf{T}^2), \quad (4.23)$$

where  $\lambda_\alpha(D)$  is a pseudo-differential operator with the symbol  $\lambda_\alpha(\xi)$  where  $\xi$  is a covariable of  $\theta = (\theta_1, \theta_2)$ .

In view of the arguments above, one may assume that  $T$  is a Fredholm operator. We shall microlocalize  $T$  in the following way. Let  $\varepsilon > 0$  and let  $\xi \in \mathbf{R}^2$ ,  $|\xi| = 1$ . We define the convex cone  $C(\xi, \varepsilon)$  and the projection  $\pi_\xi$  respectively, by

$$C(\xi, \varepsilon) := \left\{ \eta \in \mathbf{R}^2; \left| \frac{\eta}{|\eta|} - \xi \right| < \varepsilon \right\}, \quad \pi_\xi \sum_{\eta} u_\eta x^\eta / \eta! := \sum_{\eta \in C(\xi, \varepsilon) \cap \mathbf{Z}^2} u_\eta x^\eta / \eta!. \quad (4.24)$$

We define the closed subspace  $G_{w,\xi}^s(\mu)$  by

$$G_{w,\xi}^s(\mu) := \{ \pi_\xi u; u \in G_w^s(\mu) \}. \quad (4.25)$$

$G_{w,\xi}^s(\mu)$  is a Banach space with the norm of  $G_w^s(\mu)$ . We define the microlocalized Toeplitz operator  $T_\xi$  of  $T$  in the direction  $\xi$  by

$$T_\xi := \pi_\xi T \pi_\xi : G_{w,\xi}^s(\mu) \longrightarrow G_{w,\xi}^s(\mu). \quad (4.26)$$

We want to show that  $T : G_w^s(\mu) \rightarrow G_w^s(\mu)$  is a Fredholm operator if and only if, for every  $\xi \in \mathbf{R}^2$ ,  $\xi \geq 0$   $T_\xi$  is a Fredholm operator. We first show the necessity. In order to see this, we take a finite number of  $\zeta$ 's ( $|\zeta| = 1$ ,  $\zeta \in \mathbf{R}^2$ ,  $\zeta \geq 0$ ) and  $\varepsilon > 0$  in such a way that

$$\mathbf{N}^2 = \cup_{\zeta} C(\zeta, \varepsilon) \cap \mathbf{N}^2 \quad (4.27)$$

and  $C(\zeta, \varepsilon) \cap \mathbf{N}^2$  are distinct. It follows that we can decompose  $u \in G_w^s(\mu)$  into a direct sum  $u = \sum_{\zeta} \pi_{\zeta} u$ . Let  $\eta$  be one of  $\zeta$ 's in the decomposition above. Then the equation  $Tu = h$  can be written in the form

$$\pi_{\eta} T \pi_{\eta} u + \sum_{\zeta \neq \eta} \pi_{\eta} T \pi_{\zeta} u = \pi_{\eta} h, \quad \forall \eta. \quad (4.28)$$

We decompose the space  $G_w^s(\mu)$  into the direct sum  $G_w^s(\mu) = \sum_{\zeta} G_{w,\zeta}^s(\mu)$ . Similarly we decompose  $u = \sum_{\zeta} u^{\zeta}$  and  $h = \sum_{\zeta} h^{\zeta}$ , where  $u^{\zeta} := \pi_{\zeta} u$  and  $h^{\zeta} := \pi_{\zeta} h$ . If we define the vector  $\mathcal{U}$  and  $\mathcal{H}$  by  $\mathcal{U} := {}^t(u^{\zeta})_{\zeta}$  and  $\mathcal{H} := {}^t(h^{\zeta})_{\zeta}$  then (4.28) can be written in the form

$$\mathcal{A}\mathcal{U} = \mathcal{H}. \quad (4.29)$$

Here the  $(\eta, \zeta)$  component of  $\mathcal{A}$  is given by  $\pi_\eta T \pi_\zeta$ .

Because  $T$  is a Fredholm operator it follows that

$$\mathcal{A} : \bigoplus_{\zeta} G_{w,\zeta}^s(\mu) \longrightarrow \bigoplus_{\zeta} G_{w,\zeta}^s(\mu)$$

is a Fredholm operator. By setting  $u^\zeta = 0$  if  $\zeta \neq \eta$  and  $u^\eta = \pi_\eta u$  in (4.29) we have, for some compact operator  $K$  and  $c > 0$

$$\|\pi_\eta T \pi_\eta u^\eta\| + \sum_{\zeta \neq \eta} \|\pi_\zeta T \pi_\eta u^\eta\| + \|K u^\eta\| \geq c \|u^\eta\|. \quad (4.30)$$

Let  $e$  be a vector in the interior of  $C(\eta, \varepsilon)$  with integral components and let us define the shift operator  $U_e$  in the direction  $e$  by  $U_e \sum_\eta u_\eta x^\eta / \eta! := \sum_\eta u_{\eta+e} x^\eta / \eta!$ . By replacing  $u^\eta$  by  $U_e^n u^\eta$  for sufficiently large  $n$  in (4.30) we have

$$\|\pi_\eta T \pi_\eta U_e^n u^\eta\| + \sum_{\zeta \neq \eta} \|\pi_\zeta T \pi_\eta U_e^n u^\eta\| + \|K U_e^n u^\eta\| \geq c \|U_e^n u^\eta\|. \quad (4.31)$$

We note that  $\|U_e^n u^\eta\| = \|u^\eta\|$  because  $U_e$  is an isometry.

On the other hand, recalling that  $\pi_\eta$  and  $U$  commute each other we have

$$\pi_\eta T \pi_\eta U_e^n u^\eta = U_e^n \pi_\eta T \pi_\eta u^\eta + \pi_\eta [T, U_e^n] \pi_\eta u^\eta. \quad (4.32)$$

We easily see that  $[T, U_e^n]$  is a compact operator by exactly the same arguments as in the proof of (4.17). Next we shall estimate the term  $\|\pi_\zeta T \pi_\eta U_e^n u^\eta\|$  for  $\zeta \neq \eta$ . By the definition of  $U_e$  we see that the distance between the support of  $\pi_\eta U_e^n u^\eta$  and the cone  $C(\zeta, \varepsilon)$  is bounded from the below by  $c_1 n$  for some  $c_1 > 0$  independent of  $n$ . It follows from (4.22) that if  $\pi_\zeta T \pi_\eta U_e^n u^\eta$  does not vanish,  $\alpha - \beta$  in (4.22) should satisfy that  $|\alpha - \beta| \geq c_2 n$  for some  $c_2 > 0$ . Because  $\beta$  moves on a finite set and  $a_{\alpha\beta}$  tends to zero when  $\alpha$  tends to zero, it follows that the norm  $\|\pi_\zeta T \pi_\eta U_e^n u^\eta\|$  can be absorbed in  $\|u^\eta\|$  when  $n$  is sufficiently large. We fix such an integer  $n$ . Then, we have

$$\|\pi_\eta T \pi_\eta u^\eta\| + \|K' u^\eta\| \geq \delta' \|u^\eta\| \quad (4.33)$$

for some constant  $\delta' > 0$  and a compact operator  $K'$ . Because  $\eta$  is arbitrary we see that the localized operator  $\pi_\eta T \pi_\eta : G_{w,\eta}^s(\mu) \rightarrow G_{w,\eta}^s(\mu)$  is a Fredholm operator for every  $\eta$ .

We shall show the sufficiency. Let us suppose that (4.33) is valid for every  $\eta$ . We want to show that, for every  $\zeta$

$$\|\pi_\zeta T \pi_\zeta u^\zeta\| + \sum_{\eta \neq \zeta} \|\pi_\zeta T \pi_\eta u^\eta\| + \sum_{\eta} \|K u^\eta\| \geq c \sum_{\eta} \|u^\eta\|. \quad (4.34)$$

To this end, let us define the shift operators  $U_\xi$  by  $U_\xi \sum_\eta u_\eta x^\eta / \eta! := \sum_\eta u_{\eta+e(\xi)} x^\eta / \eta!$ , where  $e(\xi)$  is a vector in the interior of the cone  $C(\xi, \varepsilon)$  with integral components. Let  $n$  be an integer. We first prove the following, for some compact operator  $K$  and  $c > 0$

$$\|\pi_\zeta T \pi_\zeta U_\xi^n u^\zeta\| - \sum_{\eta \neq \zeta} \|\pi_\zeta T \pi_\eta U_\xi^n u^\eta\| + \sum_{\eta} \|K U_\xi^n u^\eta\| \geq c \sum_{\eta} \|u^\eta\|. \quad (4.35)$$

We recall that  $U_\zeta$  is an isometry. We have  $\pi_\zeta T \pi_\zeta U_\zeta^n = U_\zeta^n \pi_\zeta T \pi_\zeta + K_1$  for some compact operator  $K_1$ . Hence it follows from (4.33) that

$$\begin{aligned} \|\pi_\zeta T \pi_\zeta U_\zeta^n u^\zeta\| &\geq \|U_\zeta^n \pi_\zeta T \pi_\zeta u^\zeta\| - \|K_1 u^\zeta\| \\ &\geq \|\pi_\zeta T \pi_\zeta u^\zeta\| - \|K_1 u^\zeta\| \geq \delta' \|u^\zeta\| - \|K' u^\zeta\| - \|K_1 u^\zeta\|. \end{aligned} \quad (4.36)$$

On the other hand, we can easily see by the argument in the proof of the necessity that  $\|\pi_\zeta T \pi_\eta U_\eta^n u^\eta\|$  for  $\zeta \neq \eta$  can be absorbed in  $\|u^\eta\|$  if we take  $n$  sufficiently large. We fix such an integer  $n$ . In view of (4.33) we have (4.35).

We can remove  $U_\eta^n$  and  $U_\zeta^n$  in (4.35) by replacing  $K$  if necessary because the commutators with  $T$  and  $U_\zeta^n$ 's are compact and  $U_\zeta^n$ 's are isometries. If we take the summation with respect to  $\zeta$  in (4.35) we see that the operator  $\mathcal{A}$  is a Fredholm operator.

We define the freezed operator  $T_\eta$  by

$$T_\eta := \pi_\eta \sum_{|\beta|+(1-s)(|\alpha|-|\beta|)=m} a_{\alpha\beta} w^{\alpha-\beta} S_{\alpha-\beta} \lambda_\alpha(\eta) : G_{w,\eta}^s(\mu) \longrightarrow G_{w,\eta}^s(\mu). \quad (4.37)$$

We want to show that  $T_\eta : G_{w,\eta}^s(\mu) \rightarrow G_{w,\eta}^s(\mu)$  is a Fredholm operator if and only if  $\pi_\eta T \pi_\eta : G_{w,\eta}^s(\mu) \rightarrow G_{w,\eta}^s(\mu)$  is a Fredholm operator. Indeed, suppose that  $\pi_\eta T \pi_\eta$  is a Fredholm operator. It follows from (4.33) that

$$\|\pi_\eta T_\eta u^\eta\| + \|K' u^\eta\| \geq \delta' \|u^\eta\| - \|(T_\eta - \pi_\eta T \pi_\eta) u^\eta\|$$

for some constant  $\delta > 0$ . In view of the definition of  $\lambda_\alpha(\partial)$  in (4.22) and (4.37) we see that the operator norm of  $T_\eta - \pi_\eta T \pi_\eta$  tends to zero if  $\varepsilon \rightarrow 0$ . This implies that the term  $\|(T_\eta - \pi_\eta T \pi_\eta) u^\eta\|$  can be absorbed in  $\delta' \|u^\eta\|$ . This implies that  $T_\eta$  is a Fredholm operator. The converse part will be proved similarly.

Now our theorem is the consequence of the following theorem.

**Theorem 4.1.** *For every  $\xi$  the map  $T_\xi : G_{w,\xi}^s(\mu) \rightarrow G_{w,\xi}^s(\mu)$  is a Fredholm operator if and only if the conditions (2.8) and (2.9) are satisfied.*

The proof of this theorem is purely based on the arguments of the theory of Toeplitz operators. In order to make the presentation selfcontained, it is necessary to prepare many concepts, which would make this note considerably long. Hence we would like to omit the proof and we refer the readers to [2] for the necessary tools in proving this type of theorems.

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## References

- [1] Boutet de Monvel, L. and Guillemin V., The spectral theory of Toeplitz operators, *Annals of Mathematics Studies*, **99** Princeton University Press, 1981.
- [2] Böttcher, A. and Silbermann, B., *Analysis of Toeplitz operators*, Springer Verlag Berlin 1990.
- [3] Kashiwara, M., Kawai, T. and Sjöstrand J., On a class of linear partial differential equations whose formal solutions always converge, *Ark. für Math.*, **17** (1979), 83-91.
- [4] Komatsu, H., On the index of ordinary differential operators, *J. Fac. Sci. Univ. Tokyo*, **18** (1971), 379-398.
- [5] Komatsu, H., Linear ordinary differential equations with Gevrey coefficients, *J. Differential Eq.*, **45** (1982), 272-306.
- [6] Malgange, B., Sur les points singuliers des équations différentielles linéaires, *Enseign. Math.* **20** (1970), 146-176.
- [7] Miyake, M., On the determinant of matrices of ordinary differential operators and an index theorem, *Funk. Ekvac.*, **26** (1983), 211-230.
- [8] Miyake, M., Solvability of systems of ordinary differential equations in the space of Aronszajn and the determinant over the Weyl algebra, *Nagoya Math. J.*, **117** (1990), 207-225.
- [9] Miyake, M. and Yoshino, M., Wiener-Hopf equation and Fredholm property of the Goursat problem in Gevrey space, Preprint series No.13 (1992), College of General Education, Nagoya University.
- [10] Miyake, M. and Yoshino, M., Toeplitz operators and an index theorem for differential operators on Gevrey spaces, Preprint series No.4 (1993), College of General Education, Nagoya University.
- [11] Ramis, J.P., Dévissage Gevrey, *Asterisques* **59/60** (1978), 173-204.
- [12] Ramis, J.P., Théorèmes d'indices Gevrey pour les équations différentielles ordinaires, *Mem. Amer. Math. Soc.* **48**, (1984).